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ENVELOPES AND WEAKLY RADICALS OF SUBMODULES

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ABSTRACT. Let N be a submodule of a finitely generated module M over a Noetherian ring. A method for the computation of the submodule generated by the envelope of N is given. The relations between weakly prime submodules and their envelopes are investigated. Using these relations, a description of the weakly radical of a submodule is obtained. The results are illustrated by examples.

INTRODUCTION

Throughout this paper all rings are commutative with identity and all modules are unitary.

Let R be a ring and M be an R-module. A proper submodule P of M is said to be *primary submodule* if whenever $rm \in P$ where $r \in R$ and $m \in M$ then $m \in P$ or $r^k M \subseteq N$ for some positive integer k.

Recall that $(P : M) = \{r \in R | rM \subseteq P\}$. If P is a primary submodule of M and $p = (P : M)$, then P is called p-primary submodule (see [8]).

A *primary decomposition* of a submodule N of M is representation of N as an intersection of finitely many primary submodule of M. Such a primary decomposition $N = \cap_{i=1}^{n} Q_i$ with p_i-primary submodules Q_i is called minimal if p_i's are pairwise distinct and $Q_j \not\supseteq \cap_{i \neq j} Q_i$ for all $j = 1, \ldots, n$.

If R is a Noetherian ring and M is a finitely generated module, then any proper submodule N has a minimal primary decomposition. The first uniqueness theorem states that for such a minimal primary decomposition the set of primes $\{p_1, \ldots, p_m\}$ is uniquely defined. These primes are called the associated primes of M/N. We denote this set by $Ass(M/N)$. It is clear that for any $p \in Ass(M/N)$, $(N : M) \subseteq p$.

The prime ideals in $Ass(M/N)$ that are minimal with respect to inclusion are called the isolated primes of M/N, the remaining associated prime ideals are the embedded primes of M/N.

The second uniqueness theorem states that not only the primes but also the primary components corresponding to isolated primes, the isolated components of N in M, are uniquely defined. The other primary components, the embedded components of N in M, need not be defined uniquely. The concepts and theorems about the primary decomposition of modules can be found in chapter 9 of [10].

The radical $\sqrt{I}$ of an ideal $I \subset R$ is characterized as the the set of elements $a \in R$ such that $a^n \in I$ for some positive integer n. The concept of envelope of a submodule is the generalization of this characterization to the modules. If N is a

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submodule of an R-module M, then the envelope of N in M is defined to be the set

$$E_M(N) = \{rm : r \in R, m \in M \; and \; r^k m \in N \; for \; some \; k \in \mathbb{Z}^+\}.$$

Let $\langle E_M(N)\rangle$ be the submodule generated by the envelope. Although some methods for computing of radical of a submodule , which defined to be intersection of prime submodules containing N, are given in [7] and [9], it seems there is no description for the computation of the envelope in the literature. In section 1, we give a formula for the computation of $\langle E_M(N)\rangle$ if a minimal primary decomposition of N is known. In this section, we use extensively the concepts and results from [6].

A proper submodule N of an R-module M is called *a weakly prime submodule* if for each $m \in M$ and $a, b \in R$; $abm \in N$ implies that $am \in N$ or $bm \in N$. A proper submodule N of an R-module M is called *a weakly primary submodule* if $abm \in N$ where $a, b \in R$ and $m \in M$, then either $bm \in N$ or $a^k m \in N$ for some $k \geq 1$. The concepts of weakly prime and weakly primary submodules are introduced a few years ago and they have been studied by some authors (for example see [1], [2] and [3]). In section 2, we investigated relations between weakly prime submodules and their envelopes. We also give an example to show a conjecture given in [3] is false.

The weakly radical of a submodule N of M, denoted by $wrad_M(N)$, is defined to be the intersection of all weakly prime submodules containing N. In [2], a generalization of $\langle E_M(N)\rangle$ defined as follows: $E_0(N) = N, E_1(N) = E_M(N), E_2(N) = E_M(\langle E_M(N)\rangle)$, and for any positive integer n, it is defined $E_{n+1}(N) = E_M(\langle E_n(N)\rangle)$ inductively. $E_n(N)$ is called n-th envelope of N. Consider

$$UE_M(N) = \bigcup_{n\in\mathbb{N}} \langle E_n(N)\rangle;$$

$UE_M(N)$ is called the *union of envelopes of* N. One can easily see that $N \subseteq \langle E_n(N)\rangle \subseteq UE_M(N) \subseteq wrad_M(N) \subseteq rad_M(N)$, for any $n \in \mathbb{N}$. If $UE_M(N) = rad_M(N)$(resp. $UE_M(N) = wrad_M(N)$), then it said to be radical formula (resp. weakly radical formula) holds for N.

A submodule N is called a quasi-p-primary submodule in M, if N has a unique isolated prime p and possibly some embedded primes (see [6]). In section 3, we show that weakly radical formula hold for quasi-primary submodules. Using this, we give a method to compute weakly radical of a submodule of a Noetherian module M.

1. Envelope of Submodules

Unless otherwise stated, after this point, we assume R is a Noetherian ring, M is finitely generated R- module and N is proper submodule of M.

Lemma 1.1. *Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ be a minimal primary decomposition of N where $\sqrt{Q_i : M} = p_i$ for all $i = 1, 2, \ldots, k$. If $S = \{1, 2, \ldots, k\}$ and $\emptyset \neq T \subsetneq S$, then*

$$(\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i) \subseteq \langle E_M(N)\rangle$$

Proof. Let $n \in (\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i)$. Then there exist $r_j \in \bigcap_{i\in T} p_i$ and $m_j \in \bigcap_{i\in S\setminus T} Q_i$ such that

$$n = r_1 m_1 + r_2 m_2 + \cdots + r_s m_s$$

for some $s \in \mathbb{Z}^+$.

Since $r_j \in \bigcap_{i \in T} p_i$, ${r_j}^{k_j} M \subseteq \bigcap_{i \in T} Q_i$ for some $k_j \in \mathbb{Z}^+$. In particular, ${r_j}^{k_j} m_j \in \bigcap_{i \in T} Q_i$ for all $j = 1, 2, \cdots, s$.

Since $m_j \in \bigcap_{i \in S-T} Q_i$, ${r_j}^{k_j} m_j \in \bigcap_{i \in S-T} Q_i$ for all j. Thus we have ${r_j}^{k_j} m_j \in \bigcap_{i=1}^{k} Q_i = N$ which means that $r_j m_j \in E_M(N)$ for all j. Thus $n \in \langle E_M(N) \rangle$. $\square$

Before giving a formula for the envelope of a submodule in terms of its associated primes and primary submodules in its primary decomposition, we need some technical prerequisites.

Definition 1.2. If $f \in R$ and I is an ideal of R, then the set

$$N : I^{\infty} = \{m \in M : I^k m \subseteq N \text{ for some positive integer } k\}$$

is called the *stable quotient* of N by I in M.

Lemma 1.3. [6, Lemma 1] *Let $P \subset M$ be a primary submodule of M and $f \in R$.*

$$(i)\ \ P : \langle f \rangle^{\infty} = M \ \text{ if } \ f \in \sqrt{P : M}$$

$$(ii)\ \ P : \langle f \rangle^{\infty} = P \ \text{ if } \ f \notin \sqrt{P : M}$$

More generally, for arbitrary submodule N of M and its primary decomposition $N = \bigcap P_i$ into p_i-primary submodules P_i we get

$$(iii)\ \ N : \langle f \rangle^{\infty} = \bigcap_{f \notin p_i} P_i$$

and for arbitrary ideal I of R

$$(iv)\ \ N : I^{\infty} = \bigcap_{I \not\subset p_i} P_i$$

We can easily show that.

Lemma 1.4. *Let N be p-primary submodule of an R-module M. Then*

$$(i)\ \ N : h = N, \ \text{ if } \ h \notin p$$

$$(ii)\ \ N : h = M, \ \text{ if } \ h \in (N : M).$$

The following theorem is the main result of this section.

Theorem 1.5. *With the notation in Lemma 1.1,*

$$\langle E_M(N) \rangle = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{\emptyset \neq T \subsetneq S} (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i).$$

Proof. Let $m \in \langle E_M(N)\rangle$. Then there exist $m_j \in M, r_j \in R$ such that

$$m = r_1m_1 + r_2m_2 + \cdots + r_tm_t.$$

By the definition of $\langle E_M(N)\rangle$, $m_j \in N : \langle r_j\rangle^\infty$ for each $j = 1, 2, \ldots, t$.

For each r_j, either $r_j \in R \setminus \bigcup_{i=1}^{k} p_i$ or there is a maximal proper subset T of S such that $r_j \in \bigcap_{i\in T} p_i$.

If $r_j \in R \setminus \bigcup_{i=1}^{k} p_i$, then $N : \langle r_j\rangle^\infty = N$ by Lemma 1.3. Hence $m_j \in N$ and so $r_jm_j \in N$.

If $r_j \in \bigcap_{i\in T} p_i$, then

$$N : \langle r_j\rangle^\infty = \bigcap_{i=1}^{k}(Q_i : \langle r_j\rangle^\infty) = \bigcap_{i\in S\setminus T} Q_i$$

by Lemma 1.3. Hence

$$r_jm_j \in (\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i).$$

If $r_j \in \bigcap_{i=1}^{k} p_i = \sqrt{N : M}$, then $r_jm_j \in \sqrt{N : M}M$.

Thus we can conclude that

$$\langle E_M(N)\rangle \subseteq N + (\bigcap_{i=1}^{k} p_i)M + \sum_{\emptyset\neq T\subsetneq S} (\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i).$$

For the other side of the inclusion, Lemma 1.1 implies that

$$\sum_{\emptyset\neq T\subsetneq S} (\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i) \subseteq \langle E_M(N)\rangle.$$

Moreover N and $(\bigcap_{i=1}^{k} p_i)M = \sqrt{N : M}M$ are clearly in $\langle E_M(N)\rangle$. □

Corollary 1.6. *If N is a p-primary submodule, then*

$$\langle E_M(N)\rangle = N + pM.$$

Now we will give an application of Theorem 1.5. The computer algebra system Singular was used for the computations (see [5]).

Example 1.7. Let $R = \mathbb{Q}[x, y, z]$ and let $M = R \oplus R \oplus R$. Consider the submodule $N = \langle xz\mathbf{e}_3 - z\mathbf{e}_1, x^2\mathbf{e}_3, x^2y^3\mathbf{e}_1 + x^2y^2z\mathbf{e}_2\rangle$.

Primary decomposition of N is $N = Q_1 \cap Q_2 \cap Q_3$ where

$$Q_1 = \langle \mathbf{e}_3, z\mathbf{e}_1, y\mathbf{e}_1 + z\mathbf{e}_2, z^2\mathbf{e}_2\rangle \text{ is } \langle z\rangle - \text{primary},$$
$$Q_2 = \langle \mathbf{e}_1, \mathbf{e}_3, y^2\mathbf{e}_2\rangle \text{ is } \langle y\rangle - \text{primary and}$$
$$Q_3 = \langle x\mathbf{e}_1, x\mathbf{e}_3 - \mathbf{e}_1, x^2\mathbf{e}_2\rangle \text{ is } \langle x\rangle - \text{primary.}$$

By Theorem 1.5,

$$\langle E_M(N)\rangle = N + (p_1\cap p_2\cap p_3)M + p_1(Q_2\cap Q_3) + p_2(Q_1\cap Q_3) + p_3(Q_1\cap Q_2)$$
$$+(p_1\cap p_2)Q_3 + (p_1\cap p_3)Q_2 + (p_2\cap p_3)Q_1.$$

It is clear that $(p_1\cap p_2\cap p_3)M = \langle xyz\mathbf{e}_1, xyz\mathbf{e}_2, xyz\mathbf{e}_3\rangle$. We also get

$$\begin{aligned}
p_1(Q_2\cap Q_3) &= \langle xz\mathbf{e}_1, xz\mathbf{e}_3 - z\mathbf{e}_1, x^2y^2z\mathbf{e}_2\rangle\\
p_2(Q_1\cap Q_3) &= \langle xyz\mathbf{e}_3 - yz\mathbf{e}_1, x^2y\mathbf{e}_3, x^2y^2\mathbf{e}_1 + x^2yz\mathbf{e}_2\rangle\\
p_3(Q_1\cap Q_2) &= \langle x\mathbf{e}_3, xz\mathbf{e}_1, xy^3\mathbf{e}_1 + xy^2z\mathbf{e}_2\rangle\\
(p_1\cap p_2)Q_3 &= \langle xyz\mathbf{e}_1, xyz\mathbf{e}_3 - yz\mathbf{e}_1, x^2yz\mathbf{e}_2\rangle\\
(p_1\cap p_3)Q_2 &= \langle xz\mathbf{e}_1, xz\mathbf{e}_3, xy^2z\mathbf{e}_2\rangle\\
(p_2\cap p_3)Q_1 &= \langle xy\mathbf{e}_3, xyz\mathbf{e}_1, xy^2\mathbf{e}_1 + xyz\mathbf{e}_2, xyz^2\mathbf{e}_2\rangle
\end{aligned}$$

Thus

$$\langle E_M(N)\rangle = \langle z\mathbf{e}_1, x\mathbf{e}_3, xyz\mathbf{e}_2, xy^2\mathbf{e}_1\rangle.$$

Corollary 1.8. *If $\langle E_M(N)\rangle = N$, then each isolated component of primary decomposition of N must be prime.*

Proof. Let $N = Q_1\cap Q_2\cap\cdots\cap Q_n$ with Q_i's are p_i-primary submodules. Let Q_k be one of the isolated components of N. If Q_k were not a prime submodule, then there would be exist $x\in p_k\setminus(Q_k : M)$. Hence there exists $m\in M$ such that $xm\notin Q_k$. Since p_k is an isolated prime, we can find an element $y\in(\bigcap_{j\neq k} p_j)\setminus p_k$. Then

$$xym\in(\bigcap_{j=1}^{n} p_j)M\subseteq\langle E_M(N)\rangle = N\subseteq Q_k.$$

Since Q_k is p_k-primary and $xm\notin Q_k$, $y\in p_k$ which is a contradiction. □

In general for submodules N_1 and N_2 of a module M, $\langle E_M(N_1\cap N_2)\rangle\neq\langle E_M(N_1)\rangle\cap\langle E_M(N_2)\rangle$. We would like to give a condition for submodules under which we have the equality.

Definition 1.9. A submodule N is called a quasi-p-primary submodule in M, if N has a unique isolated prime p (and possibly embedded primes).

The following proposition is crucial for the computing primary decomposition and is quite useful for our purpose.

Proposition 1.10. [6, Proposition 1] *Assume that $L=\{p_1,\dots,p_k\}$ are the isolated primes of N. For $i,j=1,\dots,m$ take $f_i\in R$ such that $f_i\in p_j$ if $i\neq j$, but $f_i\notin p_i$ and take integers e_i such that $f_i^{e_i}N_i\subset N$.*
Then:
(i) N_i is a quasi-p_i-primary module in M.
(ii) The sets $A_i = Ass(M/Ni) = \{p\in Ass(M/N) : f_i\notin p\}$ are pairwise disjoint.
(iii) For $J := \langle f_1, f_2,\dots,f_k\rangle$ we have

$$N = (\bigcap Ni)\cap(N+JM)$$

This is a decomposition of N into quasi-primary components N_i and a component $N' := N+JM\subset M$ of lower (relative) dimension.

Theorem 1.11. *Assume that $L = \{p_1, \dots, p_k\}$ are the isolated primes of N and the minimal primary decomposition of N contains only quasi-primary components N_i for $i = 1, \dots, k$. If $\langle E_M(N)\rangle = N$, then $\langle E_M(N_i)\rangle = N_i$ for each quasi-primary component N_i. Hence*

$$\langle E_M(N)\rangle = \langle E_M\big(\bigcap_{i=1}^{k}(N_i)\big)\rangle = \bigcap_{i=1}^{k}\langle E_M(N_i)\rangle.$$

Proof. For a fixed i, let $Ass(M/N_i) = \{p_{i_1} = p_i, p_{i_2}, \dots, p_{i_{s_i}}\}$ and $p_i \subseteq p_{i_k}$ for every k and let $N_i = Q_{i_1} \cap \dots \cap Q_{i_{s_i}}$ where each Q_{i_k} is p_{i_k}-primary. By the Theorem 1.5,

$$\langle E_M(N)\rangle = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{\emptyset \neq T \subsetneq S}\big(\bigcap_{j\in T} p_{i_j}\big)\big(\bigcap_{j\in S\setminus T} Q_{i_j}\big)$$

and

$$\langle E_M(N_i)\rangle = N_i + p_iM + \sum_{T \subsetneq S_i}\big(\bigcap_{r\in T} p_{i_r}\big) \setminus \big(\bigcap_{r\in S_i\setminus T} Q_{i_r}\rangle$$

where $S_i = \{i_1, i_2, \dots, i_{s_i}\}$ and $S = \bigcup_{i=1}^{k} S_i$.

Let $x \in p_i$ and $m \in M$. Take $y = \big(\bigcap_{j\neq i} p_j\big) \setminus \big(\bigcup_{t=2}^{s_i} p_{i_t}\big)$. Then

$$yxm \in \big(\bigcap_{j=1}^{k} p_j\big)M \subseteq \langle E_M(N)\rangle \subseteq Q_{i_t}$$

for $t = 1, \dots, s_i$. Since Q_{i_t} is primary and $y \notin p_{i_t}$, $xm \in Q_{i_t}$. Hence $xm \in N_i$.

Now let $x \in \bigcap_{r\in T}, m \in \bigcap_{S_i\setminus T} Q_{i_r}$ for some $T \subsetneq S_i$. Take

$$y = \big(\bigcap_{j\neq i} p_j\big) \setminus \big(\bigcup_{t=2}^{s_i} p_{i_t}\big).$$

Then

$$yxm \in \Big[\big(\bigcap_{j\neq i} p_j\big) \cap \big(\bigcap_{r\in T} p_{i_t}\big)\Big]\big(\bigcap_{r\in S_i\setminus T} Q_{i_r}\big).$$

Since

$$\bigcap_{j\neq i} p_j = \bigcap_{j\neq i}\bigcap_{t=1}^{s_j} p_{j_t},$$

$$\Big[\big(\bigcap_{j\neq i} p_j\big) \cap \big(\bigcap_{r\in T} p_{i_t}\big)\Big]\big(\bigcap_{r\in S_i\setminus T} Q_{i_r}\big) \subseteq \langle E_M(N)\rangle \subseteq N_i.$$

Thus $yxm \in Q_{i_t}$ for $t = 1, \dots, s_i$. Since Q_{i_t} is primary and $y \notin p_{i_t}$, $xm \in Q_{i_t}$ and hence $xm \in N_i$. Therefore $\langle E_M(N_i)\rangle = N_i$ and the conclusion easily follows. □

2. Weakly Prime Submodules

In this section we investigate the relations between weakly prime submodules and their envelopes.

Lemma 2.1. *If N is a weakly prime submodule, then $\langle E_M(N)\rangle = N$.*

Proof. Let $x \in \langle E_M(N)\rangle$. Then there exist elements $r_i \in R$ and $m_i \in M$ $(1 \le i \le k)$ such that

$$x = r_1m_1 + \cdots + r_km_k \qquad with \quad r_i^{t_i}m_i \in N$$

for some $t_i \in \mathbb{Z}^+$. Since N is weakly prime, $r_i^{t_i}m_i \in N$ implies that $r_im_i \in N$ or $r_i^{t_i-1}m_i \in N$. If $r_im_i \in N$, then $x = r_1m_1 + \cdots + r_km_k \in N$. If $r_i^{t_i-1}m_i \in N$, then $r_im_i \in N$ or $r_i^{t_i-2}m_i \in N$. By the same process, $r_im_i \in N$ for all cases. Hence $x \in N$, which means that $\langle E_M(N)\rangle \subseteq N$. Other side of the inclusion is obvious. □

Theorem 2.2. *Suppose that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ where each Q_i is p_i-primary submodule with $p_1 \subset p_2 \subset \cdots \subset p_s$. If $E_M(N) = N$, then N is a weakly prime submodule.*

Proof. Since $p_1 \subset p_2 \subset \cdots \subset p_s$, by the Theorem 1.5

$$N = \langle E_M(N)\rangle = N + p_1M + \sum_{i=2}^{s} p_i\Big(\bigcap_{j=1}^{i-1} Q_j\Big).$$

Let $abm \in N$ with $a, b \in R$ and $m \in M$. Let i be the first index for which $m \notin Q_i$. Since Q_i is p_i-primary, $ab \in p_i$ and so either $a \in p_i$ or $b \in p_i$. If $i = 1$, then since $p_1M \subset \langle E_M(N)\rangle = N$, either $am \in N$ or $bm \in N$. Let $i > 1$. Since $p_i(\bigcap_{j=1}^{i-1} Q_j) \subset \langle E_M(N)\rangle = N$, either $am \in N$ or $bm \in N$. Hence N is a weakly prime submodule. □

The following conjecture is stated in [3]: Let R be a ring and M be an R-module. Then for every weakly primary submodule Q of M, $\langle E_M(Q)\rangle$ is a weakly prime submodule. Notice that they use the notation $\sqrt[nil]{Q}$ for $\langle E_M(Q)\rangle$ in [3].

The following example shows that the conjecture is false.

Example 2.3. Let $R = \mathbb{Q}[x, y]$ and let $M = R \oplus R$. Consider the submodule $N = \langle x\mathbf{e}_1 + y^3\mathbf{e}_2, x^2\mathbf{e}_1, x\mathbf{e}_2\rangle$. One can easily see that $(N : M) = \langle x^2\rangle$ and N is $\langle x\rangle$-primary submodule. Hence

$$\langle E_M(N)\rangle = N + \langle x\rangle M = \langle x\mathbf{e}_1, x\mathbf{e}_2, y^3\mathbf{e}_2\rangle$$

Then $\langle E_M(N)\rangle$ is not weakly prime submodule since $y^2(0, y) = (0, y^3) \in \langle E_M(N)\rangle$ but $y(0, y) = (0, y^2) \notin \langle E_M(N)\rangle$.

If we weaken the conditions of the conjecture as follows, then we can obtain the desired result.

Corollary 2.4. *Let R be a Noetherian ring and M be a finitely generated R-module. Then for every weakly primary submodule Q of M; if $\langle E_M(Q)\rangle = Q$, then Q is weakly prime.*

Proof. Let $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ be primary decomposition of Q with $\sqrt{Q_i : M} = p_i$ $(1 \le i \le k)$. By [3, Proposition 3.1], $p_1 \subset p_2 \subset \cdots \subset p_k$. Then Theorem 2.2 implies that Q is weakly prime submodule. □

Corollary 2.5. *Let $N = Q_1 \cap Q_2$ be a submodule of M where Q_i is p_i-primary. If $\langle E_M(N)\rangle = N$, then either Q_1 and Q_2 are both prime or N is weakly prime.*

Proof. We have two cases: $p_1 \nsubseteq p_2$ or $p_1 \subseteq p_2$. If $p_1 \nsubseteq p_2$, then both p_1 and p_2 are isolated primes. From Corollary 1.8, Q_1 and Q_2 are prime submodules. If $p_1 \subseteq p_2$, then Theorem 2.2 implies that N is weakly prime. □

Lemma 2.6. *If N is a quasi-p_1-primary submodule and $\langle E_M(N)\rangle = N$, then N can be expressed as an intersection of finitely many weakly prime submodules containing N.*

Proof. Let $Ass(M/N) = \{p_1, \ldots, p_s\}$ and $S = \{1, \ldots, s\}$. If N contains only one maximal associated prime with respect to inclusion, then its associated primes form a chain $p_1 \subset \cdots \subset p_s$. Hence N is weakly prime by Theorem 2.2.

Suppose that N has more than one maximal element. For each maximal p_j, we have a unique chain of associated primes $p_1 = p_{j_1} \subset p_{j_2} \subset \cdots \subset p_{j_t} = p_j$. Let $N_j = Q_{j_1} \cap Q_{j_2} \cdots \cap Q_{j_t}$ where $Q_{j_1} = Q_1$ and $Q_{j_t} = Q_j$. From Theorem 1.5,

$$\langle E_M(N)\rangle = N + p_1M + \sum_{T\subset S} (\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i)$$

and

$$\langle E_M(N_j)\rangle = N_j + p_1M + \sum_{i=2}^{t} p_{j_i}(\bigcap_{k=1}^{i-1} Q_{j_k}).$$

Our aim to show that $\langle E_M(N_j)\rangle = N_j$. Clearly $p_1M \subset \langle E_M(N)\rangle = N \subset N_j$. Let $B = Ass(M/N) \setminus Ass(M/N_j)$. Take $x \in p_{j_i}$ and $m \in \bigcap_{k=1}^{i-1} Q_{j_k}$. Since p_j is a maximal prime and associated primes pairwise distinct, there exists $y \in (\bigcap_{p\in B} p)\setminus p_j$. Hence

$$yxm \in (p_{j_i} \cap (\bigcap_{p\in B} p)(\bigcap_{k=1}^{i-1} Q_{j_k}) \subset \langle E_M(N)\rangle = N \subset N_j \subset Q_{j_k}.$$

Since each Q_{j_k} is p_{j_k}-primary and $y \notin p_{j_k}$, $xm \in Q_{j_k}$. Hence $xm \in N_j$. This implies $\langle E_M(N_j)\rangle = N_j$ and N_j is weakly prime by Theorem 2.2. Since $N = \cap N_j$, N is intersection of finitely many weakly prime submodules. □

Using the previous Lemma and Theorem 1.11, we can conclude the following.

Theorem 2.7. *Assume that $L = \{p_1, \ldots, p_k\}$ are the isolated primes of N and the minimal primary decomposition of N contains only quasi-primary components N_i for $i = 1, \ldots, k$. If $\langle E_M(N)\rangle = N$, then N can be expressed as the intersection of finitely many weakly prime submodules.*

Definition 2.8. A proper submodule N of an R-module M is called semiprime if whenever $r^k m \in N$ for some $r \in R, m \in M$ and natural number k, then $rm \in N$.

The question when a semiprime module can be expressed as a finite intersection of weakly prime submodules discussed in [4]. We have the following contribution to this discussion.

Lemma 2.9. *Let N be a semiprime submodule of an R-module M. Then $\langle E_M(N)\rangle = N$.*

Proof. Let $x \in \langle E_M(N)\rangle$. Then there exist elements $r_i \in R$, $m_i \in M$ $(1 \le i \le k)$ such that

$$x = r_1m_1 + \cdots + r_km_k \qquad with \quad r_i^{t_i} m_i \in N$$

for some $t_i \in \mathbb{Z}^+$. Since N is semiprime, $r_im_i \in N$ for all i. Hence $x \in N$ and $\langle E_M(N)\rangle = N$. □

Corollary 2.10. *Let R be a Noetherian ring and M be a finitely generated R-module. Each semiprime submodule N of M is intersection of weakly prime submodules, if the primary decomposition of N contains only quasi-primary components for each isolated prime of N.*

We have also the following result.

Proposition 2.11. *Let N be a weakly primary submodule of M. Then N is semiprime if and only if N is weakly prime.*

Proof. Suppose N is semiprime. Then by Lemma 2.9, $\langle E_M(N)\rangle = N$. Since N is weakly primary, N is weakly prime by Corollary 2.4.

Conversely assume N is weakly prime. Let $r \in R, m \in M$ and $r^k m \in N$ for some $k \in \mathbb{Z}^+$. $r^k m \in N$ implies that $rm \in N$ or $r^{k-1}m \in N$. If $rm \in N$, then N is semiprime. If $r^{k-1}m \in N$, then by the same process $rm \in N$. Hence for all cases N is semiprime submodule. □

3. Weakly Radical

The next two results are quiet useful for our purpose.

Lemma 3.1. [9, Lemma 2.3] *For every prime ideal p of R such that $(N : M) \subseteq p$, $(N + pM : M) = p$.*

Corollary 3.2. [9, Corollary 2.4] *$p \in Ass(M/(N+pM))$ if and only if $(N : M) \subseteq p$.*

We can generalize this results.

Lemma 3.3. *If $(N : M) = p_1$ for a prime ideal p_1, then $(\langle E_M(N)\rangle : M) = p_1$.*

Proof. Since $(N : M) = p_1$ and every associated prime of N contains $(N : M)$, N is a quasi-p_1-primary submodule. Suppose $N = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of N where each Q_i is p_i-primary. Let $S = \{1, \ldots, s\}$ and let T be a non-empty proper subset of S. Clearly, N is a quasi-p_1-primary submodule. Since all associated primes of N contain p_1, $\sqrt{N : M} = p_1$ and $(\bigcap_{i\in T} p_i)(\bigcap_{i\in S-T} Q_i) \subset p_1 M$ when $1 \in T$. Hence

$$\langle E_M(N)\rangle = N + p_1 M + \sum_{1\notin T\subsetneq S} (\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i).$$

Since $(\bigcap_{i\in S\setminus T} Q_i) \subset Q_1$ when $1 \notin T$,

$$\big((\bigcap_{i\in T} p_i)(\bigcap_{i\in S-T} Q_i)\big) : M \subseteq Q_1 : M \subseteq p_1.$$

Let

$$K = \big(N + \sum_{1\notin T\subsetneq S} (\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i)\big).$$

Then $\langle E_M(N) = K + p_1 M\rangle$ where $K : M \subseteq p_1$. By Lemma 3.1, $(\langle E_M(N)\rangle : M) = p_1$. □

The following concept is crucial in computation of radical of a submodule.

Definition 3.4. Let p be any prime ideal of R. Following [9], we let $cl_p(N)$ denote the p-closure of N, as defined by

$$cl_p(N) = \{m \in M : rm \in N \text{ for some } r \in R/p\}$$

It is clear then $cl_p(N) = \cup_{r \in R \setminus p}(N : r)$ and $N \subseteq cl_p(N)$. The most interesting case is where $(N : M) \subseteq p$. In fact, if a minimal primary decomposition of N is known, then $cl_p(N)$ can be computed as an intersection of certain primary submodule of the primary decomposition.

Lemma 3.5. *Let p be prime ideal such that $(N : M) \subseteq p$. If $N = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition with p_i-primary submodule Q_i's, then*

$$cl_p(N) = \bigcap_{p_i \subseteq p} Q_i$$

Proof. Let $r \in R \setminus p$. Then by Lemma 1.4

$$(N : r) = \bigcap_{i=1}^{s}(Q_i : r) = \big(\bigcap_{p_i \nsubseteq p}(Q_i : r)\big) \cap \big(\bigcap_{p_i \subseteq p} Q_i\big).$$

If $p_i \subseteq p$ for all i's, then we obtain the result. If $p_i \nsubseteq p$ for some i, then there exists $r_i \in (Q_i : M) \setminus p$. Let $r_0 = \prod_{p_i \nsubseteq p} r_i$. Since $r_0 \in (Q_i : M)$ for each i satisfying $p_i \nsubseteq p$, Lemma 1.4 implies

$$(N : r_0) = \big(\bigcap_{p_i \nsubseteq p}(Q_i : r_0)\big) \cap \big(\bigcap_{p_i \subseteq p} Q_i\big) = \bigcap_{p_i \subseteq p} Q_i.$$

The conclusion is obvious. □

In [9], the authors just compute $cl_p(N + pM)$ which is the only closure needed in the computation of the radical. The above lemma gives a method of computation $cl_p(N)$ for any submodule N whose primary decomposition is known.

Before defining similar concept for the computation of weakly radical, we need the following result.

Theorem 3.6. *If $(N : M) = p$ for a prime ideal p of R, then the weakly radical formula hold for N.*

Proof. Since $(N : M) = p$, N and $\langle E_M(N)\rangle$ are quasi-p-primary submodules by Lemma 3.3. Furthermore $\langle E_n(N)\rangle$ is also quasi-p-primary for any $n \in \mathbb{Z}^+$ by the same reason. Since we assumed M is a finitely generated module over a Noetherian ring R, $UE_M(N) = \langle E_k(N)\rangle$ for some $k \in \mathbb{Z}^+$. That means $\langle E_k(N)\rangle = \langle E_t(N)\rangle$ for every $t \geq k$. By Lemma 2.6, $UE_M(N)$ can be expressed as an intersection of finitely many weakly prime submodules containing N. Therefore $wrad_M(N) = UE_M(N)$. □

Definition 3.7. Let $(N : M) = p$ for a prime ideal p of R. We let $wcl_p(N)$ denote the weakly p-closure of N, as defined by $wcl_p(N) = UE_M(N)$.

At this point we would like to emphasize that the associated primes of a weakly primary submodules should form a chain according to [3, Proposition 3.1]. Since every weakly prime submodule is also weakly primary, the associated primes of the prime submodules also satisfied this property. Hence all weakly prime submodules have a unique isolated prime.

Lemma 3.8. *If P is a weakly prime submodule containing a submodule N and if p_1 is the isolated prime of P, then $p_1 \supseteq (N : M)$.*

Proof. Let P be a weakly prime submodule with the isolated prime p_1. Therefore $E_M(P) = P$ by Lemma 2.1. If $P = Q_1 \cap \cdots \cap Q_s$ is the minimal primary decomposition where each Q_i is p_i-primary, then Q_1 is p_1-prime submodule by Corollary 1.8. Hence $(N : M) \subseteq (P : M) \subseteq (Q_1 : M) = p_1$. □

Hence when computing weakly radical of a submodule, we can restrict ourself to the weakly prime submodules whose isolated primes contain $N : M$.

Given Lemma 3.1 and Theorem 3.6, $wcl_p(N + pM)$ is of particular interest. Of course, if $(N : M) \subseteq p$, $N \subseteq N + pM \subseteq UE(N + pM) = wcl_p(N + pM) = wrad_M(N+pM)$. Hence $wcl_p(N+pM)$ can be expressed as an intersection of weakly prime submodules containing N. The next theorem shows that these weakly prime submodules are minimal among the weakly prime submodules with isolated prime p containing N.

Theorem 3.9. *Let P be weakly prime submodule containing N and let p_1 be the isolated prime of P, then $wcl_{p_1}(N + p_1M) \subseteq P$.*

Proof. Since P is weakly prime submodule with the isolated prime p_1, $E_M(P) = P$ and its associated primes forms a chain $p_1 \subset p_2 \cdots \subset p_s$. Since $\langle E_M(P)\rangle = P + p_1M + \cdots$, $N + p_1M \subset P + p_1M \subset E_M(P) = P$. Since $(N + p_1M) : M = p_1$, the the weakly radical formula hold for $N + p_1M$. Then

$$wcl_{p_1}(N + p_1M) = UE_M(N + p_1M) \subset UE_M(P) = P.$$

□

Proposition 3.10. *If P_1 and P_2 are weakly primary submodules containing N with isolated primes p_1 and p_2 respectively and $p_1 \subseteq p_2$, then $wcl_{p_1}(N + p_1M) \subseteq wcl_{p_2}(N + p_2M)$.*

Proof. Since $p_1 \subseteq p_2$, $N + p_1M \subseteq N + p_2M$. Then clearly $\langle E_M(N + p_1M)\rangle \subseteq \langle E_M(N + p_2M)\rangle$. Hence the conclusion is obvious. □

If R is a Noetherian ring and $I \subset R$ is an ideal, the set of associated primes of I is the set $Ass(I) = \{P \subset R | P \,\text{prime}\, P = I : \langle b\rangle \,\text{for some}\, b \in R\}$. The set of associated primes which are minimal with respect to set inclusion is denoted by $minAss(I)$. Hence using Theorem 3.9 and Proposition 3.10 we can give the following formula for the computation of $wrad_M(N)$.

Corollary 3.11.

$$wrad_M(N) = \bigcap_{p \in minAss((N:M))} wcl_p(N + pM).$$

Now we illustrate the computation of the weakly radical of a submodule by an example. We again use the computer algebra system SINGULAR for the computations (see [5]).

Example 3.12. Let $R = \mathbb{Q}[x, y, z]$ and let $M = R \oplus R \oplus R$. Consider the submodule

$$N = \langle x^2\mathbf{e_1} + y^2\mathbf{e_2}, x^2z\mathbf{e_2}, y^3z\mathbf{e_1} + z^3\mathbf{e_3}\rangle.$$

$Ass(M/N) = \{\langle z\rangle, \langle x\rangle\}$.

$W_1 = N + \langle z\rangle M = \langle z\mathbf{e_1}, z\mathbf{e_2}, z\mathbf{e_3}, x^2\mathbf{e_1} + y^2\mathbf{e_2}\rangle$.

Since W_1 is $\langle z\rangle$-prime, $wcl_{\langle z\rangle}(W_1) = UE_M(W_1) = W_1$.

$W_2 = N + \langle x \rangle M = \langle x\mathbf{e_1}, x\mathbf{e_2}, x\mathbf{e_3}, y^2\mathbf{e_2}, y^3z\mathbf{e_1} + z^3\mathbf{e_3} \rangle$.
$Ass(M/W_2) = \{p_1 = \langle x \rangle, p_2 = \langle x, z \rangle, p_3 = \langle x, y \rangle\}$.
The primary decomposition of $W_2 = Q_1 \cap Q_2 \cap Q_3$ where
$Q_1 = \langle \mathbf{e_2}, x\mathbf{e_1}, x\mathbf{e_3}, y^3\mathbf{e_1} + z^2\mathbf{e_3} \rangle$,
$Q_2 = \langle \mathbf{e_2}, z\mathbf{e_1}, z\mathbf{e_3}, x\mathbf{e_1}, x\mathbf{e_3} \rangle$ and
$Q_3 = \langle \mathbf{e_3}, x\mathbf{e_1}, x\mathbf{e_2}, y^2\nvdash, y^2\mathbf{e_2} \rangle$. Here each Q_i is p_i-primary.
Using the Theorem 1.5, one can compute that

$$\langle E_M(W_2) \rangle = \langle y\mathbf{e_2}, x\mathbf{e_1}, x\mathbf{e_2}, x\mathbf{e_3}, y^3z\mathbf{e_1} + z^3\mathbf{e_3} \rangle.$$

In fact, $\langle E_M(\langle E_M(W_2) \rangle) \rangle = \langle E_M(W_2) \rangle$. Therefore $wcl_{\langle x \rangle}(W_2) = \langle E_M(W_2) \rangle$.
Thus

$$\begin{aligned} wrad_M(N) &= wcl_{\langle z \rangle}(W_1) \cap wcl_{\langle x \rangle}(W_2) \\ &= \langle yz\mathbf{e_2}, xz\mathbf{e_1}, xz\mathbf{e_2}, xz\mathbf{e_3}, x^2\mathbf{e_1} + y^2\mathbf{e_2}, y^3z\mathbf{e_1} + z^3\mathbf{e_3} \rangle. \end{aligned}$$

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